ON RELATIVE CONTENT AND GREEN'S LEMMA*

BY

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It has been shown† that if the line integral $\int_C x\,dy$ exists over a simple closed plane curve C, then the content of K, the interior of C, exists equal to that integral. This result may be thought of as the special case of Green's lemma

$$(1) \qquad \iint_K P_x(xy)\,dx\,dy = \int_C P(xy)\,dy$$

in which $P(xy)=x$; and it is to be noted that here C does not need to be rectifiable.

In the present paper a definition of relative content is given which makes it possible to prove that if P and P_x are subject to certain conditions, the content of K, relative to a certain non-additive function of rectangles derived from P, exists equal to the double integral on the left of (1) and also equal to the line integral on the right of (1) whenever that integral exists. This result includes as a special case the form of Green's lemma for rectifiable C obtained by Gross,‡ except that in our result P_x is deliberately restricted to be properly Riemann integrable instead of summable. In the last section sufficient conditions for the existence of the line integral are given which yield Green's lemma for an important case in which C does not need to be rectifiable.

1. **Definitions and elementary theorems.** Let $\mathfrak{P}$ denote a class of partitions Π of the rectangle R_0: $a \leqq x \leqq b$, $c \leqq y \leqq d$, such that (1) each partition Π is formed by dividing R_0 into vertical and horizontal strips; and (2) the (greatest) lower bound of the norms of the partitions Π of $\mathfrak{P}$ is zero; here by the norm of a partition Π of $\mathfrak{P}$ is meant the (least) upper bound of the lengths of the diagonals of the rectangles of which Π consists.

Moreover let $f(R)$ be a function (not necessarily single-valued) defined for every rectangle R: $x' \leqq x \leqq x''$, $y' \leqq y \leqq y''$ lying in R_0. Also, if K_1 and K_2 are any two sets in R_0, let $\epsilon(K_1, K_2)=1$ if K_1 and K_2 have at least one point

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† H. L. Smith, these Transactions, vol. 27, p. 498.

‡ Wm. Gross, Monatshefte für Mathematik und Physik, vol. 26, p. 70. See also Van Vleck, Annals of Mathematics, (2), vol. 22, p. 226; Bray, Annals of Mathematics, (2), vol. 26, p. 278.

in common, and let $\epsilon(K_1, K_2)=0$ if K_1 and K_2 have no point in common. Finally let $\Delta\Pi$ denote any one of the rectangles of which Π consists and let $N\Pi$ denote the norm of Π.

Then if K is a set in R_0 and

$$\lim_{N\Pi=0} \sum_{\Delta\Pi} f(\Delta\Pi)\epsilon(K, \Delta\Pi)$$

exists, it is called the outer content of K relative to f. In the above the summation is naturally over all $\Delta\Pi$. If

$$\lim_{N\Pi=0} \sum_{\Delta\Pi} f(\Delta\Pi)[1-\epsilon(R_0-K, \Delta\Pi)]$$

exists, it is called the inner content of K relative to f. If both the outer and the inner contents of K relative to f exist and are equal, their common value is called the content of K relative to f.

The outer content of K relative to f exists *absolutely* if it not only exists relative to f but also relative to $|f|$. Similar definitions are given to the absolute existence of the inner content and of the content itself.

A set K is *squarable* relative to f if its content exists and equals zero; it is absolutely squarable relative to f if its content exists absolutely relative to f and is zero.

We mention the following obvious theorem:

THEOREM 1. *If a set K is absolutely squarable every subset of K is absolutely squarable.*

We now prove

THEOREM 2. *If the boundary of a set K, entirely interior to R_0, is absolutely squarable relative to f, then the content of K exists if either the inner content or the outer content exists relative to f.*

For

$$\sum_{\Delta\Pi} f(\Delta\Pi)\epsilon(K, \Delta\Pi) = \sum_{\Delta\Pi} f(\Delta\Pi)[1-\epsilon(R_0-K, \Delta\Pi)] + T(\Pi),$$

where

$$T(\Pi) = \sum_{\Delta\Pi} f(\Delta\Pi)\epsilon(K, \Delta\Pi)\epsilon(R_0-K, \Delta\Pi).$$

But

$$|T(\Pi)| \leqq \sum_{\Delta\Pi} |f(\Delta\Pi)|\epsilon(K_b, \Delta\Pi),$$

where K_b denotes the boundary of K. Hence $\lim_{N\Pi=0} T(\Pi)=0$, from which the theorem follows.

THEOREM 3. *If K_1 and K_2 are both interior to R_0 and have no points in common and if each has inner content (f) and one of their boundaries is squarable absolutely (f), then the inner content (f) of K_1+K_2 exists equal to the sum of the inner contents (f) of K_1 and K_2.*

For suppose K_{1b}, the boundary of K_1, is squarable absolutely (f). Then, if we set $K=K_1+K_2$,

$$\sum_{\Delta\Pi} f(\Delta\Pi)[1-\epsilon(R_0-K,\Delta\Pi)] = \sum_{\Delta\Pi} f(\Delta\Pi)[1-\epsilon(R_0-K_1,\Delta\Pi)] + \sum_{\Delta\Pi} f(\Delta\Pi)[(1-\epsilon(R_0-K_2,\Delta\Pi)] + T(\Pi),$$

where

$$T(\Pi) = \sum_{\Delta\Pi} f(\Delta\Pi)\epsilon(K_1,\Delta\Pi)\epsilon(K_2,\Delta\Pi)[1-\epsilon(R_0-K,\Delta\Pi)].$$

But

$$|T(\Pi)| \leqq \sum_{\Delta\Pi} |f(\Delta\Pi)|\epsilon(K_{1b},\Delta\Pi).$$

Hence $\lim_{N\Pi=0} T(\Pi)=0$, from which the theorem follows.

THEOREM 4. *If K_1 and K_2 are interior to R_0 and have no points in common and if K_1 and K_2 each have outer content (f) and one of their boundaries is squarable absolutely (f), then the outer content (f) of K_1+K_2 exists equal to the sum of the outer contents (f) of K_1 and K_2.*

For suppose K_{1b}, the boundary of K_1, is absolutely squarable (f). Then

$$\sum_{\Delta\Pi} f(\Delta\Pi)\epsilon(K_1+K_2,\Delta\Pi) = \sum_{\Delta\Pi} f(\Delta\Pi)\epsilon(K_1,\Delta\Pi) + \sum_{\Delta\Pi} f(\Delta\Pi)\epsilon(K_2,\Delta\Pi) - T(\Pi),$$

where

$$T(\Pi) = \sum_{\Delta\Pi} f(\Delta\Pi)\epsilon(K_1,\Delta\Pi)\epsilon(K_2,\Delta\Pi).$$

But

$$|T(\Pi)| \leqq \sum_{\Delta\Pi} |f(\Delta\Pi)|\epsilon(K_{1b},\Delta\Pi).$$

Hence $\lim_{N\Pi=0} T(\Pi)=0$; from which the theorem follows.

Theorems 2, 3 and 4 now give

THEOREM 5. *If K_1 and K_2 are interior to R_0 and have no points in common and if K_1 and K_2 each have content (f) and one of their boundaries is absolutely squarable (f), then the content of K_1+K_2 exists (f) and equals the sum of the contents (f) of K_1 and K_2.*

We shall need the following special case of Theorem 5.

THEOREM 6. *If K is the interior of a simple closed curve C which is interior to R_0 and K has inner content (f) and C is absolutely squarable (f), then K and $C+K$ each have content (f) and their contents are equal.*

2. **On the existence of relative content.** Let $P(xy)$ and $Q(xy)$ be defined on R_0. Then let two (multiply-valued) functions $P^{(x)}(R)$ and $Q^{(y)}(R)$ be defined as follows:

$$P^{(x)}(R) = P(x''y) - P(x'y), \qquad y' \leqq y \leqq y'';$$
$$Q^{(y)}(R) = Q(xy'') - Q(xy'), \qquad x'' \leqq x \leqq x';$$

where R is the rectangle $x' \leqq x \leqq x''$, $y' \leqq y \leqq y''$, which is assumed to be in R_0. In this section we shall be interested in content relative to $P^{(x)}Q^{(y)}$ in the special case where $Q(xy) = y$.

THEOREM 7. *If P_x, the first partial derivative of P with respect to x, exists on K, the interior of a simple closed squarable curve C in R_0, and if P_x is bounded and integrable on K, then the inner content of K relative to $P^{(x)}y^{(y)}$ exists absolutely and equals $\iint_K P_x\,dxdy$.*

To prove this, note that by the mean value theorem

$$\sum_{\Delta\Pi} P^{(x)}(\Delta\Pi)y^{(y)}(\Delta\Pi)[1 - \epsilon(R_0 - K, \Delta\Pi)] = \sum_{\Delta\Pi} P_x(\zeta^{\Delta\Pi})\Delta\Pi[1 - \epsilon(R_0 - K, \Delta\Pi)],$$

where $\zeta^{\Delta\Pi}$ is a point (xy) in $\Delta\Pi$. But

$$\sum_{\Delta\Pi} P_x(\zeta^{\Delta\Pi})\Delta\Pi[1 - \epsilon(R_0 - K, \Delta\Pi)] = \sum_{\Delta\Pi} P_x(\zeta^{\Delta\Pi})(K\cdot\Delta\Pi)\epsilon(K, \Delta\Pi) - T(\Pi),$$

where $K\cdot\Delta\Pi$ denotes the set of points common to K and $\Delta\Pi$ and where

$$T(\Pi) = \sum_{\Delta\Pi} P_x(\zeta^{\Delta\Pi})(K\cdot\Delta\Pi)\epsilon(R_0 - K, \Delta\Pi)\epsilon(K, \Delta\Pi);$$

here $\zeta^{\Delta\Pi}$ has already been defined if $\epsilon(R_0 - K, \Delta\Pi) = 0$ and is defined as any point (xy) in $K\cdot\Delta\Pi$ otherwise. But then

$$\lim_{N\Pi=0} \sum_{\Delta\Pi} P_x(\zeta^{\Delta\Pi})(K\cdot\Delta\Pi)\epsilon(K\cdot\Delta\Pi) = \iint_K P_x\,dxdy$$

and since

$$|T(\Pi)| \leqq N\sum_{\Delta\Pi}(\Delta\Pi)\epsilon(C, \Delta\Pi),$$

where N is the least upper bound of $|P_x|$ on K, it follows that

$$\lim_{N\Pi} T(\Pi) = 0.$$

This proves the theorem.

THEOREM 8. *If P_x exists and is bounded and integrable on R_0 and if C is a simple closed squarable curve interior to R_0, then C is absolutely squarable $(P^{(x)}y^{(v)})$.*

For

$$\sum_{\Delta\Pi} | P^{(x)}(\Delta\Pi)y^{(v)}(\Delta\Pi) | \epsilon(C, \Delta\Pi) = \sum_{\Delta\Pi} | P_x(\zeta^{\Delta\Pi}) | (\Delta\Pi)\epsilon(C, \Delta\Pi)$$

$$\leqq N \cdot \sum_{\Delta\Pi}(\Delta\Pi)\epsilon(C, \Delta\Pi).$$

Hence

$$\lim_{N\Pi} \sum_{\Delta\Pi} | P^{(x)}(\Delta\Pi)y^{(v)}(\Delta\Pi) | \epsilon(C, \Delta\Pi) = 0,$$

which is the theorem.

THEOREM 9. *If K is a region bounded by a simple closed squarable curve C and P_x exists and is bounded and integrable on R_0, then K and $K+C$ each has content equal to $\iint_K P_x\,dxdy$ relative to $P^{(x)}y^{(v)}$.*

This follows from Theorems 6, 7 and 8.

It has now become necessary to make an additional assumption concerning $\mathfrak{P}$. We say a partition Π of R_0 is of type (A) if* $x^{(x)}(\Delta\Pi)$ is constant for all $\Delta\Pi$ of Π and if $y^{(v)}(\Delta\Pi) \leqq x^{(x)}(\Delta\Pi)$ for every $\Delta\Pi$ of Π. A partition Π of R_0 is of type (B) if it can be obtained from a partition of type (A) by subdividing some or all of the cells of that partition into at most three parts each by means of vertical lines. We assume throughout the remainder of the paper that $\mathfrak{P}$ consists of all partitions of R_0 of type (A) or type (B).

We are now in a position to prove

LEMMA 1. *If C is a simple closed rectifiable curve interior to R_0, then*

$$\sum_{\Delta\Pi} y^{(v)}(\Delta\Pi)\epsilon(C, \Delta\Pi) \leqq 6 \cdot 2^{1/2}C,$$

for every Π with norm sufficiently small, where C denotes the length of C.

To prove this we note that if r is less than one-half the diameter of C,

$$C_{(r)} \leqq 2rC, \dagger$$

where $C_{(r)}$ denotes the outer content of the set of all points distant by not more than r from C. But if also $r = 2^{2/1}x^{(x)}(\Delta\Pi)$, and Π is of type (A),

* Naturally we are here considering only partitions of R which satisfy condition (1) of §1.

† This follows from a similar inequality for a simple arc stated by Gross, Monatshefte, vol. 29, p. 177. The proof given by Gross is incomplete; a correct proof is to be found in the author's Chicago dissertation.

$$\sum_{\Delta\Pi}(\Delta\Pi)\epsilon(C,\,\Delta\Pi) \leqq C_{(r)} \leqq 2rC = 2\cdot 2^{1/2}x^{(x)}(\Delta\Pi)C.$$

Hence since $\Delta\Pi = x^{(x)}(\Delta\Pi)y^{(y)}(\Delta\Pi)$ and $x^{(x)}(\Delta\Pi)$ is constant,

$$\sum_{\Delta\Pi} y^{(y)}(\Delta\Pi)\epsilon(C,\,\Delta\Pi) \leqq 2\cdot 2^{1/2}C,$$

which proves the result for type (A); from this the result easily follows for type (B).

THEOREM 10. *If P is continuous in x uniformly as to (xy) on R_0 and C is a simple closed rectifiable curve interior to R_0, then C is absolutely squarable $(P^{(x)}y^{(y)})$.*

For

$$\sum_{\Delta\Pi} |\, P^{(x)}(\Delta\Pi)y^{(y)}(\Delta\Pi)\,|\, \epsilon(C,\,\Delta\Pi) \leqq a(\Pi)\sum_{\Delta\Pi} y^{(y)}(\Delta\Pi)\epsilon(C,\,\Delta\Pi) \leqq a(\Pi)6\cdot 2^{1/2}C,$$

where $a(\Pi)$ is the largest value of $|P^{(x)}(\Delta\Pi)|$ for all $\Delta\Pi$ of Π. But on account of the uniform continuity,

$$\lim_{N\Pi} a(\Pi) = 0,$$

from which the theorem follows.

3. **On the x-linear extension of P. Its uniform continuity.** So far we have been considering the function $P(xy)$ as defined on the entire rectangle R_0. We now suppose that $P(xy)$ is defined on a closed set S interior to R_0 and show how to extend its definition to the entire plane. To this end let (x_0y_0) be a point not in S. Let $\underline{x}_0$ be the lower bound of all x' such that for $x' \leqq x \leqq x_0$ the point (xy_0) is not in S. Since S is closed it is clear that if $\underline{x}_0$ is finite the point $(\underline{x}_0y_0)$ is in S. Similarly let $\bar{x}_0$ be the upper bound of all x'' such that for $x_0 \leqq x \leqq x''$ the point (xy_0) is not in S; if $\bar{x}_0$ is finite, the point $(\bar{x}_0y_0)$ is in S. We now define $P(x_0y_0)$ as follows:

$$P(x_0y_0) = P(\underline{x}_0y_0) + (x_0 - \underline{x}_0)P(\underline{x}_0\bar{x}_0,\, y_0),$$

if $\underline{x}_0$, $\bar{x}_0$ are both finite, where

$$P(\underline{x}_0\bar{x}_0,\, y_0) = [P(\bar{x}_0y_0) - P(\underline{x}_0y_0)]/[\bar{x}_0 - \underline{x}_0].$$

We also make the following definitions: $P(x_0y_0) = P(\bar{x}_0y_0)$ if $\underline{x}_0$ only is infinite; $P(x_0y_0) = P(\underline{x}_0y_0)$ if $\bar{x}_0$ only is infinite; $P(x_0y_0) = 0$ if both $\underline{x}_0$ and $\bar{x}_0$ are infinite. We call the function whose definition has been thus extended the x-linear extension of P.

THEOREM 11. *If P is defined on and interior to a simple closed curve C, is continuous on C and has a bounded first partial derivative P_x on K, the interior of C, then the x-linear extension of P is continuous as to x uniformly as to (xy) on R_0.*

To prove this we note that since P is continuous on C it is uniformly continuous there, that is, there is a system $(d_e' \mid e)$ such that

$$| P(x_1y_1) - P(x_2y_2) | \leqq e/3$$

for (x_1y_1), (x_2y_2) on C and $[(x_1-x_2)^2+(y_1-y_2)^2]^{1/2} \leqq d_e'$.

We say two points $(x_0'y_0)$, $(x_0''y_0)$ are of the first kind if they are both on C; of the second kind if (xy_0) is inside C for every x between x_0' and x_0''; of the third kind if (xy_0) is outside C for every x between x_0' and x_0''.

Consider first a pair $(x_0'y_0)$, $(x_0''y_0)$ of the first kind. It follows from the above that for such a pair

$$| P(x_0'y_0) - P(x_0''y_0) | \leqq e/3$$

if $|x_0' - x_0''| \leqq d_e'$.

Next consider a pair of the second kind. Here by the mean value theorem

$$| P(x_0'y_0) - P(x_0''y_0) | \leqq | P_x(x'''y_0) | \; | x_0' - x_0'' | \leqq N | x_0' - x_0'' | \leqq e/3,$$

if $|x_0' - x_0''| \leqq d_e''$, where d_e'' is the smaller of d_e' and $e/(3N)$, N being the least upper bound on P_x on K, and x''' is between x_0' and x_0''.

Consider next a pair of the third kind. In this case there is a pair of the first kind and also of the third kind* $(x_0^{\mathrm{I}}y_0)$, $(x_0^{\mathrm{II}}y_0)$ such that $x_0^{\mathrm{I}} \leqq x_0'$, $x_0'' \leqq x_0^{\mathrm{II}}$. But then by definition of x-linear extension

$$| P(x_0'y_0) - P(x_0''y_0) | = |P(x_0^{\mathrm{I}}y_0) - P(x_0^{\mathrm{II}}y_0)| \, |x_0' - x_0'| / |x_0^{\mathrm{I}} - x_0^{\mathrm{II}}| \leqq e/3$$

for $|x_0' - x_0''| \leqq d_e$, where d_e is the smaller of d_e'' and $ed_e'/(6M)$, M being the least upper bound of P on C.

We now consider an arbitrary pair of points $(x_0'y_0)$, $(x_0''y_0)$ in R_0 such that $|x_0' - x_0''| \leqq d_e$. The interval $(x_0'x_0'')$ can be broken up into at most three sub-intervals each of which with y_0 gives rise to a pair of points either of the first or second or third kind. Hence

$$| P(x_0'y_0) - P(x_0''y_0) | \leqq e/3 + e/3 + e/3 = e$$

for $|x_0' - x_0''| \leqq d_e$, which proves the desired theorem.

Theorems 6, 7, 10, 11 now give

* This is true unless $P(x_0'y_0) = P(x_0''y_0)$, in which case the result is obvious.

THEOREM 12. *If P is defined on a simple closed rectifiable curve C and its interior K, is continuous on C and possesses a bounded integrable first partial derivative P_x on K, then K and $K+C$ each have content equal to $\int\int P_x dxdy$ relative to $P_1^{(x)}y^{(y)}$ and C is absolutely squarable relative to $P_1^{(x)}y^{(y)}$, where P_1 is the x-linear extension of P.*

4. **The generalized Green's lemma.** It is the purpose of this section to prove

THEOREM 13. *If P is defined on R_0, and C, a simple closed curve interior to R_0, is absolutely squarable $(P^{(x)}y^{(y)})$, if K, the interior of C, has inner content $(P^{(x)}y^{(y)})$, and if, moreover, the integral $\int_C Pdy$ exists,* then*

$$\int_C Pdy = \text{cont}_{P^{(x)}y^{(y)}}K = \text{cont}_{P^{(x)}y^{(y)}}(K + C).$$

Let

$$C: \qquad x = \phi(t), \ y = \psi(t) \qquad (0 \leqq t \leqq 1)$$

be parametric equations of C such that as t varies from 0 to 1, C is described in the positive sense. For brevity write

$$P_0(t) = P[\phi(t), \psi(t)].$$

Now let e be a fixed positive number and π_0 a fixed partition of (01) into intervals $\Delta\pi_0$,

$$\pi_0: \qquad t_0(=0), t_1, \cdots, t_{n-1}, t_n(=1),$$

and suppose π_0 is such that if $\pi F \pi_0$,† that is, if π is a partition obtained by subdividing some or all of the intervals of π_0, then

$$\left| \int_C Pdy - S_\pi^0 P_0 \Delta\psi \right| \leqq \frac{e}{4},$$

where

$$S_\pi^0 P_0\Delta\psi = \sum_{\Delta\pi} \tfrac{1}{2}\{P_0(\overline{\Delta\pi}) + P_0(\underline{\Delta\pi})\}\psi(\Delta\pi).$$

Next let us form a partition Π_e of $\mathfrak{P}$ by dividing R_0 into horizontal strips $\rho_1, \cdots, \rho_h$ closed and non-overlapping (except for boundary points) and also into equal vertical strips $\sigma_1, \cdots, \sigma_k$ of the same character in such a way that

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* It is sufficient to assume the integral exists in the weak sense; that is all that is actually used. (Cf. the author's paper cited above.)

† Loc. cit., p. 492.

(1) the width of each horizontal strip is at most equal to the common width of the vertical strips;

(2) each of the points $(\phi(t_i), \psi(t_i))$ which corresponds to a division point t_i of π_0 is on the common boundary of two adjacent horizontal strips;

(3) the inequality

$$\sum_{\Delta\Pi} | P^{(x)}(\Delta\Pi) | y^{(v)}(\Delta\Pi)\epsilon(C, \Delta\Pi) \leqq \frac{e}{2}$$

holds for every partition $\Pi F \Pi_\epsilon$;

(4) the inequality

$$\left| \operatorname{cont}_{P^{(x)}y^{(v)}} K - \sum_{\Delta\Pi} P^{(x)}(\Delta\Pi) y^{(v)}(\Delta\Pi)[1 - \epsilon(R_0 - K, \Delta\Pi)] \right| \leqq \frac{e}{4}$$

holds for every partition $\Pi F \Pi_e$.

Let us now consider the intersection K_r of K with the interior of ρ_r. Since it is a region (that is, set of inner points), it can be resolved (uniquely) into a finite or denumerably infinite number of connected regions:

$$K_r = Q_{r1} + Q_{r2} + Q_{r3} + \cdots .$$

We say a region Q_{ri} is of the first kind if its boundary contains an arc of C which has points of intersection with both the upper and the lower boundaries of ρ_r; otherwise Q_{ri} is of the second kind. It is easily shown that for given r there are but a finite number of Q_{ri} of the first kind; suppose the notation so chosen that they are $Q_{r1}, \cdots, Q_{ri_r}$. Let Q_r be the sum of the Q_{ri} of the second kind. Then

$$K_r = Q_r + \sum_{i=1}^{i_r} Q_{ri},$$

where all the Q_{ri} are of the first kind.

It is now easily shown that since $Q_{ri}(i=1, \cdots, i_r)$ is connected, its boundary consists of (1) an arc a'_{ri} of C lying entirely within ρ_r except for its end points, of which the first* lies on the lower boundary of ρ_r and the second on the upper boundary of ρ_r; (2) an arc a''_{ri} of the same character except that its first and second end points are respectively on the upper and lower boundaries of ρ_r; (3) a finite or denumerably infinite number of arcs of C each with its end points on the same boundary line of ρ_r; (4) a finite or denumerably infinite number of segments of the upper and lower boundaries of ρ_r.

* The first end point is the one which corresponds to the smaller value of t.

We next form a certain partition π_1 of (01). To this end let I'_{ri}, I''_{ri} be the t-intervals corresponding to arcs a'_{ri}, a''_{ri}, respectively. If a division point t_i of π_0 is not an end point of some I'_{ri} or I''_{ri}, it is an end point of some I_i which is the t-interval corresponding to an arc of C which lies entirely in some strip ρ_r and has its end points on the same horizontal boundary line of that strip. Now take π_1 as the partition whose points of division are the end points of the intervals I'_{ri}, I''_{ri} and existent intervals I_r.

It can now be proved that if $\Delta\pi_1$ is an interval of π_1, then $\psi(\Delta\pi_1)=\psi(\overline{\Delta\pi_1})-\psi(\underline{\Delta\pi_1})=0$ unless $\Delta\pi_1$ is an I'_{ri} or an I''_{ri}. For then $\Delta\pi_1$ is either (1) an I_r, or (2) between an I_r and an I'_{ri} or I''_{ri}, or (3) between two intervals of types I'_{ri}, I''_{ri}. In case (1), $\psi(\Delta\pi_1)=0$ obviously. The same also holds in case (2); for otherwise $\Delta\pi_1$ would correspond to an arc of C with end points on different horizontal boundary lines of ρ_r. But then this arc would contain an arc lying entirely in some ρ_r and with its end points on different horizontal boundary lines of that ρ_r and would therefore correspond to an I'_{ri} or to an I''_{ri}, that is, $\Delta\pi_1$ would contain some I'_{ri} or some I''_{ri}, contrary to hypothesis. The case (3) is similar, and the conclusion is established.

From what has just been proved, it follows that

$$
\begin{aligned}
S^0_{\pi_1}P_0\Delta\psi &= \sum_{ri}\tfrac{1}{2}\{P_0(\bar{I}'_{ri}) + P_0(\underline{I}'_{ri})\}\psi(I'_{ri}) \\
&\quad + \sum_{ri}\tfrac{1}{2}\{P_0(\bar{I}''_{ri}) + P_0(\underline{I}''_{ri})\}\psi(I''_{ri}). \qquad (1)
\end{aligned}
$$

But

$$
\begin{aligned}
\psi(\underline{I}'_{ri}) &= \psi(\bar{I}''_{ri}) = \underline{y}_r, \text{ say}, \\
\psi(\bar{I}'_{ri}) &= \psi(\underline{I}''_{ri}) = \bar{y}_r, \text{ say},
\end{aligned}
$$

$$
\psi(I'_{ri}) = -\psi(I''_{ri}) = \Delta y_r, \text{ say}.
$$

Moreover let us write

$$
\begin{aligned}
\phi(\bar{I}'_{ri}) &= \bar{x}'_{ri}, \quad \phi(\underline{I}'_{ri}) = \underline{x}'_{ri}, \\
\phi(\bar{I}''_{ri}) &= \bar{x}''_{ri}, \quad \phi(\underline{I}''_{ri}) = \underline{x}''_{ri}.
\end{aligned}
$$

Then (1) may be written

$$
\begin{aligned}
S_{\pi_1}{}^0P\,\Delta\psi &= \sum_r \Delta y_r \sum_{i=1}^{i_r} \frac{1}{2}\{P(\bar{x}'_{ri}\bar{y}_r) - P(\underline{x}''_{ri}\bar{y}_r)\} \\
&\quad + \sum_r \Delta y_r \sum_{i=1}^{i_r} \frac{1}{2}\{P(\underline{x}'_{ri}\underline{y}_r) - P(\bar{x}''_{ri}\underline{y}_r)\}. \qquad (2)
\end{aligned}
$$

Now let R'_{ri}, R''_{ri} be the rectangles

$$R'_{ri}: \qquad \underline{x}''_{ri} \leqq x \leqq \bar{x}'_{ri}, \quad \underline{y}_r \leqq y \leqq \bar{y}_r\,;$$

$$R''_{ri}: \qquad \bar{x}''_{ri} \leqq x \leqq \underline{x}'_{ri}, \quad \underline{y}_r \leqq y \leqq \bar{y}_r.$$

Also let $\bar{\xi}'_{rsi}$, $\underline{\xi}'_{rsi}$ be respectively the upper and lower bounds of values of x for points (xy) in the (existent) rectangle $\sigma_s \cdot R'_{ri}$. Then

$$[P(\bar{x}'_{ri}\bar{y}_r) - P(\underline{x}''_{ri}\bar{y}_r)]\Delta y_r = \sum_s [P(\bar{\xi}'_{rsi}\bar{y}_r) - P(\underline{\xi}'_{rsi}\bar{y}_r)]\Delta y_r. \tag{3}$$

Therefore

$$\Big|\mathrm{cont}_{P(x),y(v)}K - \sum_r \sum_i [P(\bar{x}'_{ri}\bar{y}_r) - P(\underline{x}''_{ri}\bar{y}_r)]\Delta y_r\Big|$$

$$\leqq \Big|\mathrm{cont}_{P(x),y(v)}K - \sum_r \sum_i \sum_s [P(\bar{\xi}'_{rsi}\bar{y}_r) - P(\underline{\xi}'_{rsi}\bar{y}_r)]$$

$$\cdot [1 - \epsilon(R_0 - Q_{ri},\ \ \sigma_s \cdot R'_{ri})]\Delta y_r\Big| + \sum_r \sum_i \sum_s \Big| P(\bar{\xi}'_{rsi}\bar{y}_i)$$

$$- P(\underline{\xi}'_{rsi}\bar{y}_i)\Big| \cdot \epsilon(R_0 - Q_{ri},\ \sigma_s \cdot R'_{ri}) \cdot \Delta y_r \leqq \frac{e}{4} + \frac{e}{2} = \frac{3}{4}e.$$

Similarly

$$\Big|\mathrm{cont}_{P(x),y(v)}K - \sum_r \sum_i [P(\underline{x}'_{ri}\underline{y}_r) - P(\bar{x}''_{ri}\underline{y}_r)]\Delta y_r\Big| \leqq \frac{3}{4}e,$$

so that by (4)

$$|\ \mathrm{cont}_{P(x),y(v)}K - S_{\pi_1}{}^0 P_0 \Delta\psi\ | \leqq \tfrac{3}{4}e. \tag{4}$$

But

$$\Big|\int_C P dy - S_{\pi_1}{}^0 P_0 \Delta\psi\Big| \leqq \tfrac{1}{4}e. \tag{5}$$

From (4) and (5) the theorem follows, since e is arbitrary.

5. **On the existence of the line integral of Green's lemma.** Let the curve C and its parametric representation be as above with the additional requirement that the representation be one-to-one for $0 \leqq t < 1$. Let $P(xy)$ be defined on R_0. We seek sufficient conditions that $\int_C P_0(t)d\psi(t)$ exist, where P_0 is as above. The first such condition is given by

THEOREM 14. *If P is continuous on C, the integral $\int_C P_0(t)d\psi(t)$ exists if $\psi(t)$ is of limited variation, in particular if C is rectifiable.*

For then $P_0(t)$ is continuous and the theorem follows from a well known theorem on the Stieltjes integral.

We next obtain a condition less restrictive on C; to this end we first prove two lemmas.

LEMMA 2. *In order that*

$$\underset{\pi}{\mathrm{L}} \sum_{\Delta\pi}(O_{\Delta\pi}P_0) \,|\, \psi(\Delta\pi) \,| = 0$$

it is sufficient that

(i) *P satisfy the Lipschitz condition*

$$|\, P(x_1y_1) - P(x_2y_2) \,| \leqq A \,|\, x_1 - x_2 \,| + B \,|\, y_1 - y_2 \,|$$

for every pair of points (x_1y_1), (x_2y_2) *on* C;

(ii) $$\underset{\pi}{\mathrm{L}}\, B \sum_{\Delta\pi}(O_{\Delta\pi}\psi) \,|\, \psi(\Delta\pi) \,| = 0\,;$$

and

(iii)* $$\underset{\pi}{\mathrm{L}}\, A \sum_{\Delta\pi}(O_{\Delta\pi}\phi) \,|\, \psi(\Delta\pi) \,| = 0.$$

To prove this note that for every partition π of (01) there is a system $(t_{1\pi}^{\Delta\pi}, t_{2\pi}^{\Delta\pi} \,|\Delta\pi, \pi)$ where $t_{1\pi}^{\Delta\pi}$, $t_{2\pi}^{\Delta\pi}$ are in $\Delta\pi$, such that

$$\begin{aligned} 0 \leqq O_{\Delta\pi}P_0 &\leqq 2\,[P_0(t_{1\pi}^{\Delta\pi}) - P_0(t_{2\pi}^{\Delta\pi})] \\ &\leqq 2A \,|\, \phi(t_{1\pi}^{\Delta\pi}) - \phi(t_{2\pi}^{\Delta\pi}) \,| + 2B \,|\, \psi(t_{1\pi}^{\Delta\pi}) - \psi(t_{2\pi}^{\Delta\pi}) \,| \\ &\leqq 2[A\,O_{\Delta\pi}\phi + B O_{\Delta\pi}\psi]. \end{aligned}$$

Hence

$$\begin{aligned} 0 &\leqq \sum_{\Delta\pi}(O_{\Delta\pi}P_0) \,|\, \psi(\Delta\pi) \,| \\ &\leqq \left[A \sum_{\Delta\pi} (O_{\Delta\pi}\phi) \,|\, \psi(\Delta\pi) \,| + B \sum_{\Delta\pi}(O_{\Delta\pi}\psi) \,|\, \psi(\Delta\pi) \,| \right]. \end{aligned}$$

Therefore

$$\underset{\pi}{\mathrm{L}} \sum_{\Delta\pi}(O_{\Delta\pi}P_0) \,|\, \psi(\Delta\pi) \,| = 0,$$

as was to be proved.

LEMMA 3. *The condition*

$$\underset{\pi}{\mathrm{L}} \sum_{\Delta\pi}(O_{\Delta\pi}P_0) \,|\, \psi(\Delta\pi) \,| = 0$$

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* The condition (iii) is equivalent to the same condition with A omitted if $A \neq 0$; but it is desired to include the case when P is independent of y, in which case A may be taken to be 0; the condition is then satisfied for all ψ. A similar remark applies to (ii).

is sufficient for the existence of $\int_0^1 P_0(t)d\psi(t)$ *provided that* P *is continuous on* C *and that* $P(x'y) \leqq P(x''y)$ *for every pair of points* $(x'y)$, $(x''y)$ *on* C *such that* $x' < x''$.

For then the functions $P_0(t)$, $\psi(t)$ satisfy one of the sufficient conditions of Corollary 1, p. 505 of the author's paper cited above.

We can now state the desired condition as

THEOREM 15. *The integral* $\int_0^1 P_0(t)d\psi(t)$ *exists if*
(i) $P_x(xy)$ *exists and is less in absolute value than a fixed number* N *on* R_0, *and for each* y, $P_x(xy)$ *is a continuous function of* x;
(ii) P, P_x *satisfy the Lipschitz conditions*

$$| P(xy_1) - P(xy_2) | \leqq C | y_1 - y_2 | ,$$

$$| P_x(xy_1) - P_x(xy_2) | \leqq D | y_1 - y_2 |$$

for every pair of points (xy_1), (xy_2) *on* R_0;

(iii) $$\underset{\pi}{\mathrm{L}}(C + D) \sum_{\Delta\pi}(O_{\Delta\pi}\psi) | \psi(\Delta\pi) | = 0 ;$$

(iv) $$\underset{\pi}{\mathrm{L}} N \sum_{\Delta\pi}(O_{\Delta\pi}\phi) | \psi(\Delta\pi) | = 0.$$

To prove this let us form the functions

$$P'(xy) = P(ay) + \frac{1}{2}\int_a^x \{ | P_x(uy) | + P_x(uy) \} du,$$

$$P''(xy) = \frac{1}{2}\int_a^x \{ | P_x(uy) | - P_x(uy) \} du.$$

Clearly

(6) $$P(xy) = P'(xy) - P''(xy)$$

and

(7) $$P'(x'y) \leqq P'(x''y), \quad P''(x'y) \leqq P''(x''y) \qquad (x' < x'').$$

Now consider the function $P'(xy)$. We have

(8) $$| P'(x_1y) - P'(x_2y) | = \frac{1}{2}\left| \int_{x_1}^{x_2} \{ | P_x(uy) | + P_x(uy) \} du \right| \leqq N | x_1 - x_2 | .$$

Also

$$|P'(xy_1) - P'(xy_2)| \leqq |P(ay_1) - P(ay_2)| + \frac{1}{2}\left|\int_a^x \{ |P_x(uy_1)| - |P_x(uy_2)| + P_x(uy_1) - P_x(uy_2)\} du\right|$$

$$(9) \qquad \leqq |P(ay_1) - P(ay_2)| + \int_a^x \left| P_x(uy_1) - P_x(uy_2)\right| du$$

$$\leqq C|y_1 - y_2| + \int_a^x D|y_1 - y_2| du$$

$$\leqq C|y_1 - y_2| + (x-a)D|y_1 - y_2|$$

$$\leqq E|y_1 - y_2|,$$

where $E=C+(b-a)D$.

From (8) and (9) we get

$$(10) \qquad |P'(x_1y_1) - P'(x_2y_2)| \leqq |P'(x_1y_1) - P'(x_1y_2)| + |P'(x_1y_2) - P'(x_2y_2)| \leqq N|x_1 - x_2| + E|y_1 - y_2|.$$

If we write

$$P_0'(t) = P'[\phi(t), \psi(t)],$$

we see, from (10) and the hypothesis, that the conditions of Lemma 2 are satisfied* and hence

$$(11) \qquad \underset{\pi}{\mathrm{L}} \sum_{\Delta\pi}(O_{\Delta\pi}P_0')\psi(\Delta\pi) = 0.$$

But (11) and (7) show (Lemma 3) that $\int_0^1 P_0'(t)d\psi(t)$ exists. Similarly if $P_0''(t)=P''[\phi(t), \psi(t)]$, it can be shown that $\int_0^1 P_0''(t)d\psi(t)$ exists. Hence, by (6), $\int_0^1 P_0(t)d\psi(t)$ exists and equals $\int_0^1 P_0'(t)d\psi(t) - \int_0^1 P_0''(t)d\psi(t)$ and the theorem is proved.

6. **Two special cases.** We can now state two important special cases of Green's lemma. The first is given by

THEOREM 16. *If $P(xy)$ is defined and continuous on a simple closed rectifiable curve C and is defined and possesses a bounded integrable partial derivative $P_x(xy)$ on K, the interior of C, then $\int_C P(xy)dy$ exists and*

$$\int_C P(xy)dy = \iint_K P_x(xy)dxdy.\dagger$$

* The continuity of P in x and y together follows from (i) and (ii) which imply respectively that P is continuous in x for every y and in y uniformly as to x.

† This is the result obtained by Gross, except that he assumes $P(xy)$ to be summable instead of Riemann integrable.

This theorem follows from Theorems 12, 13, 14.

The second is given by

THEOREM 17. *If $P(xy)$ is defined on R_0 and possesses a partial derivative $P_x(xy)$ on the interior of R_0 and if C is a simple closed squarable curve interior to R_0, and K is its interior, then*

$$\int_C P(xy)dy = \int\int_K P_x(xy)dxdy,$$

provided $\int_C P(xy)dy$ exists and $P_x(xy)$ is bounded and integrable on R_0, in particular, provided

(i) *$P_x(xy)$ is, for each y, a continuous function of x;*

(ii) *P and P_x satisfy the Lipschitz conditions*

$$| P(xy_1) - P(xy_2) | \leqq C | y_1 - y_2 | ,$$

$$| P_x(xy_1) - P_x(xy_2) | \leqq D | y_1 - y_2 | ,$$

for every pair of points (xy_1), (xy_2) in R_0;

(iii) $$\mathop{\mathrm{L}}_{\pi}(C + D) \sum_{\Delta\pi}(O_{\Delta\pi}\psi) | \psi(\Delta\pi) | = 0 ;$$

(iv) $$\mathop{\mathrm{L}}_{\pi} N \sum_{\Delta\pi}(O_{\Delta\pi}\phi) | \psi(\Delta\pi) | = 0,$$

where N is the upper bound of $|P_x|$ on R_0.

This theorem follows from Theorems 8, 9, 13, 15, since the hypothesis implies the continuity in x and y together of P and P_x (see first footnote, p. 418).

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